NIDS Session 3

$\boxed{2}$

$$a_{ij} = \int_0^{c_i} l_j(x)\,dx \qquad\qquad s = 2$$

$$(c_1, c_2)$$

$$b_i = \int_0^1 l_i(x)\,dx$$

$$l_1(x) = \frac{x - c_2}{c_1 - c_2} \qquad l_2(x) = \frac{x - c_1}{c_2 - c_1}$$

$$a_{11} \int_0^{c_1} l_1(x)\,dx = \int_0^{c_1} \frac{x - c_2}{c_1 - c_2}\,dx = \frac{1}{2}\frac{(x - c_2)^2}{c_1 - c_2}\Big|_0^{c_1}$$

$$= \frac{1}{2}\left((c_1 - c_2) - \frac{c_2^2}{(c_1 - c_2)}\right)$$

$$= \frac{1}{2}\left(\frac{c_1^2 - 2c_1c_2}{(c_1 - c_2)}\right)$$

$$a_{21} = \frac{c_2^2}{2(c_2 - c_1)} \qquad a_{12} = \frac{c_1^2}{2(c_1 - c_2)}$$

$$a_{22} = \frac{1}{2}\left(\frac{c_2^2 - 2c_1c_2}{c_2 - c_1}\right)$$

$$b_1 = \frac{1 - 2c_2}{2(c_1 - c_2)} \qquad b_2 = \frac{1 - 2c_1}{2(c_2 - c_1)}$$

Order 1: $\sum_{i=1}^{s} b_i = 1 \longrightarrow \frac{1}{2(c_1 - c_2)}\left(-2(c_2 - c_1)\right) = 1$

Order 2: $\sum_{i,j=1}^{s} b_i a_{ij} = \frac{1}{2} \longrightarrow \sum_{i=1}^{s} b_i \underbrace{\sum_{j=1}^{s} a_{ij}}_{= c_i} = \sum_{i=1}^{s} b_i c_i$

$= \frac{1}{2}$

Order 3: $\sum_{i,j,k} b_i a_{ij} a_{ik} = \frac{1}{3} \longrightarrow \sum_{i=1}^{s} b_i \sum_{j} a_{ij} \sum_{k} a_{ik}$

$\sum_{i,j,k} b_i a_{ij} a_{jk} = \frac{1}{6}$

$= \sum_{i=1}^{s} b_i c_i^2$

$= \frac{1}{2}(c_1 + c_2) - c_1 c_2$

$\longrightarrow \frac{1}{2}(c_1 + c_2) - c_1 c_2$

Order 4: $\sum_{i,j,k,l} b_i a_{ij} a_{ik} a_{il} = \frac{1}{4}$

$\sum_{i,j,k,l} b_i a_{ij} a_{jk} a_{il} = \frac{1}{8}$

$\sum_{i,j,k,l} b_i a_{ij} a_{jk} a_{jl} = \frac{1}{12}$

$\sum_{i,j,k,l} b_i a_{ij} a_{jk} a_{kl} = \frac{1}{24}$

$$\begin{cases} c_1^2 - 2c_1^2 c_2 + c_1 c_2 - 2 c_1 c_2^2 + c_2^2 = \frac{1}{2} \\ c_1^2 - 2c_1^2 c_2 - 2 c_1 c_2^2 + c_2^2 = \frac{1}{3} \end{cases}$$

$c_1 = \frac{3 - \sqrt{3}}{6} \qquad c_2 = \frac{3 + \sqrt{3}}{6}$ Gauss points for $s = 2$

Exercise 3.2:

Let c_1, c_2 be distinct collocation points

Lagrange polynomials:

$$l_1(\tau) = \frac{\tau - c_2}{c_1 - c_2} \qquad l_2(\tau) = \frac{\tau - c_1}{c_2 - c_1}$$

Compute a_{ij} and b_i as a function of c_1, c_2

$$a_{ij} = \int_0^{c_i} l_j(\tau)\, d\tau \qquad b_i = \int_0^1 l_i(\tau)\, d\tau$$

We obtain

$$a_{11} = \frac{c_1}{2}\left(\frac{c_1 - 2c_2}{c_1 - c_2}\right) \qquad a_{12} = \frac{c_1^2}{2(c_1 - c_2)}$$

$$a_{21} = \frac{c_2^2}{2(c_2 - c_1)} \qquad a_{22} = \frac{c_2}{2}\left(\frac{c_2 - 2c_1}{c_2 - c_1}\right)$$

$$b_1 = \frac{1}{2}\left(\frac{1 - 2c_2}{c_1 - c_2}\right) \qquad b_2 = \frac{1}{2}\left(\frac{1 - 2c_1}{c_2 - c_1}\right)$$

Order conditions:

Order 1: $\sum_{i=1}^{s} b_i = 1$ (1)

Order 2: $\sum_{i,j=1}^{s} b_i a_{ij} = 1/2$ (2)

Order 3 : $\sum_{i,j,k=1}^{s} b_i a_{ij} a_{ik} = 1/3$ (3.1)

$$\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} = 1/6 \qquad (3.2)$$

Order 4 : $\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{ik} a_{il} = 1/4$ (4.1)

$$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{il} = 1/8 \qquad (4.2)$$

$$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{jl} = 1/12 \qquad (4.3)$$

$$\sum_{i,j,k,l=1}^{s} b_i a_{ij} a_{jk} a_{kl} = 1/24 \qquad (4.4)$$

Let's verify the order conditions:

Order 1 : $\sum_{i=1}^{s} b_i = \frac{1}{2}\left(\frac{1-2c_2}{c_1-c_2} + \frac{(2c_1-1)}{c_1-c_2} \right)$

$$= 1$$

$\rightarrow$ Order at least 1

Order 2:

$$\sum_{i,j} b_i a_{ij} = \sum_{i=1}^{s} b_i \sum_{j=1}^{s} a_{ij} = \sum_{i=1}^{s} b_i c_i$$

$$= \frac{1}{2}\left(\left(\frac{1-2c_2}{c_1-c_2}\right) c_1 + \left(\frac{2c_1-1}{c_1-c_2}\right) c_2\right) = 1/2$$

$\rightarrow$ Order at least 2 (this is expected since a collocation method has order at least s)

Order 3:

$$(3.1) = \sum_{i,j,k} b_i a_{ij} a_{ik} = \sum_i b_i \sum_j a_{ij} \sum_k a_{ik}$$

$$= \sum_{i=1}^{s} b_i c_i^2$$

$$= \frac{1}{2(c_1-c_2)}\left((1-2c_2) c_1^2 + (2c_1-1) c_2^2\right)$$

$$= \frac{1}{2(c_1-c_2)}\left(c_1^2 - c_2^2 - 2c_1c_2(c_1-c_2)\right)$$

$$= \frac{1}{2}(c_1+c_2) - c_1c_2$$

This gives us the first non trivial condition:

$$\frac{1}{2}(c_1 + c_2) - c_1 c_2 = \frac{1}{3}$$

$$(3.2) = \sum_{i,j,k} b_i a_{ij} a_{jk} = \sum_i b_i \sum_j a_{ij} \sum_k a_{jk}$$

$$= \sum_i b_i \sum_j a_{ij} c_j$$

$$= \frac{1}{2}\left(\frac{1-2c_2}{c_1 - c_2}\right)\left[\frac{c_1^2}{2}\left(\frac{c_1 - 2c_2}{c_1 - c_2}\right) + \frac{c_1^2 c_2}{2(c_1 - c_2)}\right]$$

$$+ \frac{1}{2}\left(\frac{2c_1 - 1}{c_1 - c_2}\right)\left[\frac{c_2^2 c_1}{2(c_2 - c_1)} + \frac{c_2^2}{2}\left(\frac{c_2 - 2c_1}{c_2 - c_1}\right)\right]$$

$$= \frac{1}{4}\frac{(1-2c_2)c_1^2}{(c_1 - c_2)} + \frac{1}{4}\frac{(2c_1 - 1)c_2^2}{(c_1 - c_2)}$$

$$= \frac{1}{4}\frac{1}{(c_1 - c_2)}\left(c_1^2 - c_2^2 - 2c_1c_2(c_1 - c_2)\right)$$

$$= \frac{1}{4}\left(c_1 + c_2 - 2c_1c_2\right)$$

Enforcing (3.2) leads to

$$\frac{1}{4}\left(c_1 + c_2 - 2c_1c_2 \right) = \frac{1}{6}$$

$$\frac{1}{2}\left(c_1 + c_2 \right) - c_1c_2 = \frac{1}{3}$$ which is the same condition as (3.1)

Order 4 :

$$(4.1) = \sum_{i,j,k,l} b_i a_{ij} a_{ik} a_{il} = \sum_{i=1}^{s} b_i c_i^3$$

$$= \frac{1}{2(c_1 - c_2)} \left((1 - 2c_2) c_1^3 + (2c_1 - 1) c_2^3 \right)$$

$$= \frac{1}{2(c_1 - c_2)} \left(c_1^3 - c_2^3 - 2c_1c_2 (c_1^2 - c_2^2) \right)$$

Note $c_1^3 - c_2^3 = (c_1 - c_2)(c_1^2 + c_1c_2 + c_2^2)$

Hence, (4.1) becomes

$$\frac{1}{2}\left(c_1^2 + c_1c_2 + c_2^2 - 2c_1c_2 (c_1 + c_2) \right)$$

$$= \frac{1}{2}\left(c_1^2 - 2c_1^2c_2 + c_1c_2 - 2c_1c_2^2 + c_2^2 \right)$$

Enforcing (4.1), we obtain the condition

$$c_1^2 - 2c_1^2 c_2 + c_1 c_2 - 2c_1 c_2^2 + c_2^2 = \frac{1}{2}$$

In a similar manner for (4.2), (4.3) and (4.4) we obtain two independent conditions

$$c_1^2 - 2c_1^2 c_2 + c_1 c_2 - 2c_1 c_2^2 + c_2^2 = 1/2 \qquad (1)'$$
$$c_1^2 - 2c_1^2 c_2 \qquad\qquad - 2c_1 c_2^2 + c_2^2 = 1/3 \qquad (2)'$$

In addition to the condition we had for order 3

$$\frac{1}{2}(c_1 + c_2) - c_1 c_2 = \frac{1}{3} \qquad (3)'$$

Solution :

$$(1)' - (2)' \Rightarrow \qquad c_1 c_2 = \frac{1}{6}$$

$$(3)' \Rightarrow \qquad \frac{1}{2}(c_1 + c_2) = \frac{1}{3} + c_1 c_2 = \frac{1}{2}$$

$$\Leftrightarrow \qquad c_1 + c_2 = 1$$

$$c_2 = 1 - c_1$$

$$(2)' \Rightarrow \qquad c_1^2 - 2c_1^2(1-c_1) - 2c_1(1-c_1)^2 + (1-c_1)^2 = 1/3$$

$$c_1^3(2-2) + c_1^2(1-2+4+1) + c_1(-2-2) + 1 = 1/3$$

$$4c_1^2 - 4c_1 + 1 = 1/3$$

$$4c_1^2 - 4c_1 + \frac{2}{3} = 0$$

$$\Delta = 16 - 16 \times \frac{2}{3} = \frac{16}{3}$$

roots : $$\frac{4 \pm 4/\sqrt{3}}{8} = \frac{1 \pm 1/\sqrt{3}}{2} = \frac{3 \pm \sqrt{3}}{6}$$

Since $$c_2 = 1 - c_1 = \frac{3 \mp \sqrt{3}}{6}$$

$c_1 = \dfrac{3 - \sqrt{3}}{6}$ $\quad c_2 = \dfrac{3 + \sqrt{3}}{6}$ as noted in the correction

$\boxed{4}$ i) a)

Given nodes x_i, $i = 1, \dots, n$ to be the zeros of

$$p_n(x) = \underbrace{\frac{d^n}{dx^n}\Big(\underbrace{x^n(1-x)^n}_{\mathbb{P}^{2n}}\Big)}_{\mathbb{P}^n}$$

$$p_m^{(\alpha,\beta)}(x) = \frac{1}{e_m^{(\alpha,\beta)}\, w^{(\alpha,\beta)}(x)} \frac{d^m}{dx^m}\Big(w^{(\alpha,\beta)}(x)(1+x)^m(1-x)^m\Big)$$

$$\frac{d^m}{dx^m}\Big(w^{(\alpha,\beta)}(x)\,\underline{(1+x)^m}(1-x)^m\Big) = e_m^{(\alpha,\beta)} w^{(\alpha,\beta)}(x)\, p_m^{(\alpha,\beta)}(x)$$

$\alpha = \beta = 0 \qquad w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$

$\hookrightarrow \qquad w^{(0,0)}(x) = 1$

Substitute x with $2x-1$

$$\underline{2^{2n}} \frac{d^n}{dx^n}\left(x^n(1-x)^n\right) = e_m^{(0,0)}\, \tilde{w}^{(0,0)}(x)\, \tilde{p}_m^{(0,0)}(x)$$

$$\tilde{w}^{(0,0)}(x) = w^{(0,0)}(2x-1)$$

$$\tilde{p}_m^{(0,0)}(x) = p_m^{(0,0)}(2x-1)$$

$$p_n(x) = \frac{d^n}{dx^n}\left(x^n(1-x)^n\right) = \underbrace{\frac{e_n^{(0,0)}}{2^{2n}}}_{\tilde{e}_n^{(0,0)}}\, \tilde{w}^{(0,0)}\, \tilde{p}_n^{(0,0)}(x)$$

b) We want to show that

$$\int_0^1 p_n(x)\, q(x)\, dx = 0 \qquad \forall q \in \mathbb{P}^{n-1}$$

$$\begin{cases} p_n(x) = \tilde{e}_n^{(0,0)} \underbrace{\tilde{\omega}^{(0,0)}(x)}_{1} \tilde{p}_n^{(0,0)}(x) \\ q(x) = \sum_{k=0}^{n-1} \alpha_k \tilde{p}_k^{(0,0)}(x) \end{cases}$$

$$\tilde{e}_n^{(0,0)} \int_0^1 \sum_{k=0}^{n-1} \tilde{p}_n^{(0,0)}(x) \alpha_k \tilde{p}_k^{(0,0)}(x)\, dx$$

$$= \sum_{k=1}^{n-1} e_n^{(0,0)} \alpha_k \underbrace{\int_0^1 \tilde{p}_n(x) \tilde{p}_k(x)\, dx}_{0 \text{ because } k \leq n-1} = 0$$

c) $$p_n(x) = \frac{d^n}{dx^n}\left(x^n (1-x)^n\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}}(x^n) \frac{d^k}{dx^k}\left((1-x)^n\right)$$

$$= \sum_{k=0}^{n} \underbrace{\frac{(n!)^3}{(k!)^2 ((n-k)!)^2}}_{\alpha_k} (-1)^k x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k}$$

$p_n(0) = \alpha_0 \neq 0$

$p_n(1) = \alpha_n \neq 0$

d) Let's assume by contradiction that $\exists i$ such that

i) $c_i = c_{i+1}$

$$p_n(x) = (x - c_i)^2 \underbrace{\prod_{\substack{k \neq i \\ k \neq i+1}} (x - c_k)}_{\mathbb{P}^{n-2}} = (x - c_i)^2 \, \omega(x)$$

$$\omega(x) = \prod_{\substack{k \neq i \\ k \neq i+1}} (x - c_k)$$

From question b) that $\displaystyle\int_0^1 p_n(x)\, q(x)\, dx = 0 \quad \forall q \in \mathbb{P}^{n-1}$

Let's choose $q(x) = \omega(x) \in \mathbb{P}^{n-2}$

$$\int_0^1 p_n(x)\, q(x)\, dx = \int_0^1 \underbrace{(x - c_i)^2 \, \omega(x)^2}_{\geq 0} dx > 0 \quad \text{contradiction}$$

Thus all the c_i are distinct

ii) Let's now assume $\exists\, c_i < 0$

$p_n(x) = (x - c_i)\, \omega(x) \qquad \omega(x) \in \mathbb{P}^{n-1}$

Let's pick $q(x) = \omega(x)$

$$\int_0^1 p_n(x)\, q(x)\, dx = \int_0^1 \underbrace{(x - c_i)}_{\geq 0} \underbrace{\omega(x)^2}_{\geq 0} dx > 0 \quad \text{contradiction}$$

e) We want to show that

$$\int_0^1 q(x)\,dx = \sum_{i=1}^{n} b_i\, q(c_i) \qquad \forall q \in \mathbb{P}^{2n-1}$$

Let $q(x) \in \mathbb{P}^{2n-1}$

$$q(x) = \underbrace{t(x)}_{\in \mathbb{P}^{n-1}} \underbrace{p_n(x)}_{\in \mathbb{P}^{n}} + \underbrace{r(x)}_{\in \mathbb{P}^{n-1}}$$

$$\int_0^1 q(x)\,dx = \underbrace{\int_0^1 t(x)\,p_n(x)\,dx}_{=0} + \int_0^1 r(x)\,dx = \int_0^1 r(x)\,dx$$

(see question b))

And from exercise $\boxed{3}$

$$\int_0^1 r(x)\,dx = \sum_{i=1}^{n} b_i\, r(c_i)$$

Finally, note that

$$q(c_i) = t(c_i)\underbrace{p_n(c_i)}_{=0} + r(c_i) = r(c_i)$$

because the c_i are the zeros of p_n

Hence,

$$\int_0^1 q(x)\,dx = \sum_{i=1}^{n} b_i\, r(c_i) = \sum_{i=1}^{n} b_i\, q(c_i) \quad \text{as required}$$

f) Consider $\ell_i(x) \in \mathbb{P}^{n-1} \Rightarrow \ell_i^2(x) \in \mathbb{P}^{2n-2}$

l_i^2 is exactly integrated with the Gauss rule

$$0 < \int_0^1 l_i^2(x)\, dx = \sum_{j=1}^{n} b_j\, l_i^2(c_j) = b_i$$

Thus, $b_i > 0$

(ii) a) Note that

$$p_n(x) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{x^{n-1}(1-x)^{n}}{} \right) = \underbrace{\frac{e_{n-1}^{(1,0)}}{2^{2n-1}}}_{=\tilde{e}_{n-1}^{(1,0)}} \tilde{\omega}^{(1,0)}(x)\, \tilde{p}_{n-1}^{(1,0)}(x)$$

$$\omega^{(1,0)}(x) = (1-x)$$

$$\left(\begin{array}{l} n = n-1 \\ (\alpha, \beta) = (1,0) \end{array} \right)$$

b) Similar argument

c) - Leibniz formula and evaluate at c_1

- $p_n(x) = \tilde{e}_{n-1}^{(1,0)}\, \tilde{\omega}^{(1,0)}(x)\, \tilde{p}_{n-1}^{(1,0)}(x)$

$$p_n(1) = e_{n-1}^{(1,0)} \underbrace{\tilde{\omega}^{(1,0)}(1)}_{=0} \tilde{p}_{n-1}^{(1,0)}(1) = 0$$

d) Similar argument

e)

f)

(iii) a) $p_n(x) = \dfrac{d^{n-2}}{dx^{n-2}}\left(x^{n-1}(1-x)^{n-1} \right)$

This hints that $\alpha = \beta = 1$

$(\omega^{(1,1)}(x) = (1-x)(1+x))$

$$= \tilde{e}_{n-2}^{(1,1)}\, \tilde{\omega}^{(1,1)}(x)\, \tilde{p}_{n-2}^{(1,1)}(x)$$

All remaining questions are similar, except for question f)

b) $\ell_i(x) \in \mathbb{P}^{n-1} \Rightarrow \ell_i^2(x) \in \mathbb{P}^{2n-2}$

But $\int_0^1 q(x)\,dx = \sum_{i=1}^{n} b_i\, q(c_i) \quad \forall q \in \mathbb{P}^{2n-3}$

We cannot apply the same argument

Instead, consider $\displaystyle \tilde{\ell}_i(x) := \underbrace{\frac{c_i(1-c_i)}{x(1-x)}\,\ell_i(x)}_{\in \mathbb{P}^{n-3}}$ $\qquad 1 < i < n$

$\Rightarrow \tilde{\ell}_i^2(x) \in \mathbb{P}^{2n-6}$

$$0 < \int_0^1 \underbrace{\frac{\overbrace{x(1-x)}^{\geq 0}}{c_i(1-c_i)}\,\tilde{\ell}_i^2(x)}_{\in \mathbb{P}^{2n-4}}\,dx = \sum_{j=1}^{n} b_j\,\frac{c_j(1-c_j)}{c_i(1-c_i)}\,\tilde{\ell}_i^2(c_j)$$

$$= b_i$$

$\Rightarrow b_i > 0 \qquad i = 2, \dots, n-1$

Finally, consider $\displaystyle \tilde{\ell}_1(x) = \frac{1}{(1-x)}\,\ell_1(x)$

$$\tilde{\ell}_n(x) = \frac{1}{x}\,\ell_n(x)$$

We verify that

$$0 < \int_0^1 (1-x)\,\tilde{\ell}_1^2(x)\,dx = b_1$$

$$0 < \int_0^1 x\,\tilde{\ell}_n^2(x)\,dx = b_n$$